Approximation and Finite Elements in Isogeometric Problems

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Outline

1. Introduction
   - Automated design process
   - Partial Differential Equations
   - Galerkin method
   - Isoparametric concept
2. Finite Element Method
3. Isogeometric Analysis
   - CAD basis functions
   - Analysis
4. Numerical examples
   - Numerical example (1)
   - Numerical example (2)
   - Numerical example (3)

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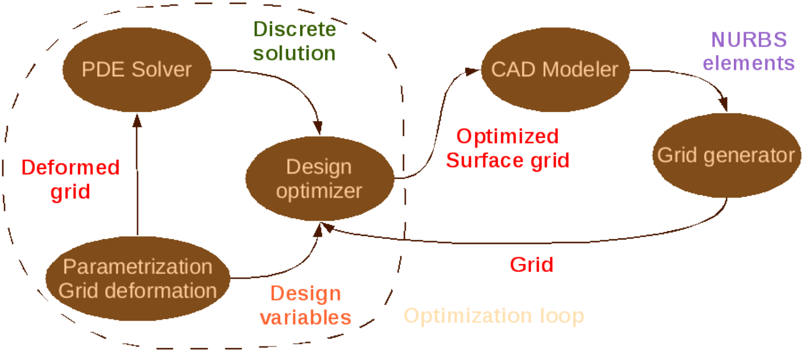
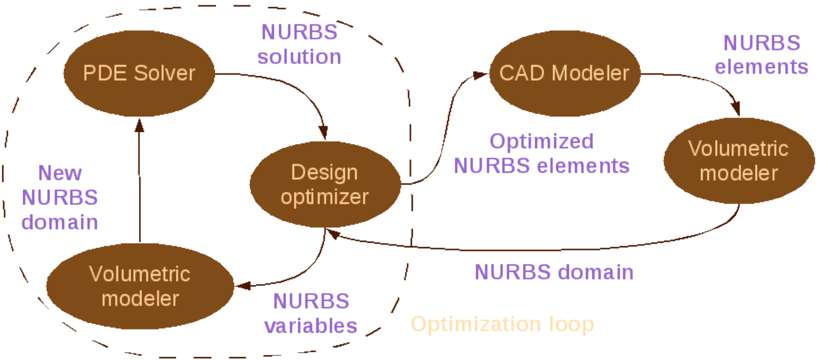
CURRENT AUTOMATED DESIGN LOOP



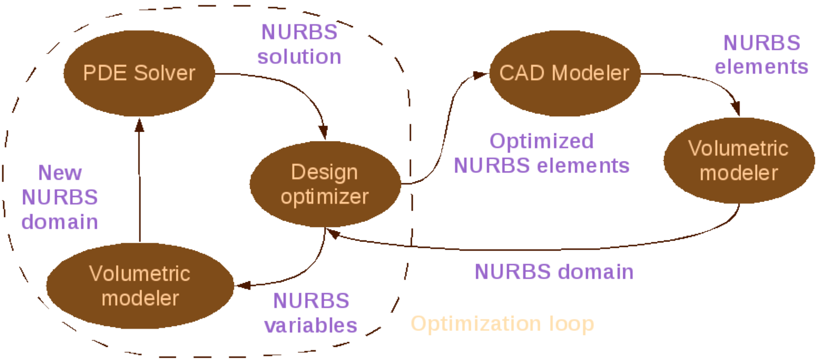
- Various tools each being complex and belonging to a specific field, e.g.:
  - CAD tools;
  - simulation tools...

ISOGEOMETRIC AUTOMATED DESIGN LOOP



- Loop entirely built on a *CAD basis*;
- all the processes can be *merged*.

ISOGEOMETRIC AUTOMATED DESIGN LOOP



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PARTIAL DIFFERENTIAL EQUATIONS

- Assuming $\boldsymbol{z} = [\boldsymbol{x}, t] = [z_1, \dots, z_n]^T$, $a_{i,j} = a_{i,j}(\boldsymbol{z}), \dots$
- the form of a *generic second-order PDE* is:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial z_i}\left(a_{ij}\frac{\partial u}{\partial z_j}\right)+\sum_{i=1}^{n}\left(\frac{\partial}{\partial z_i}(b_i u) + c_i \frac{\partial u}{\partial z_i}\right)+a_0 u = f, \ \forall \boldsymbol{z} \in \mathcal{O},$$

- equipped with adequate *boundary conditions*.

Notice!

The *strong* or *classical solution* of this problem is a solution $u(z)$ which satisfies all the equations.
It is *not possible* to find a solution in general.

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THERMAL CONDUCTION PROBLEM

Strong form of the problem

Find $u : \bar{\Omega} \to \mathbb{R}$ so that

$$\begin{cases} \nabla \left(\kappa \left(\boldsymbol{x} \right) \nabla u \left(\boldsymbol{x} \right) \right) = f \left(\boldsymbol{x} \right), & \forall \boldsymbol{x} \in \Omega, \text{ given } f : \Omega \to \mathbb{R} \\ u \left(\boldsymbol{x} \right) = g_D \left(\boldsymbol{x} \right), & \forall \boldsymbol{x} \in \Gamma_D, \ g_D : \Gamma_D \to \mathbb{R} \\ \kappa \left(\boldsymbol{x} \right) \dfrac{\partial u}{\partial \boldsymbol{\nu}} \left(\boldsymbol{x} \right) = g_N \left(\boldsymbol{x} \right), & \forall \boldsymbol{x} \in \Gamma_N, \ g_N : \Gamma_N \to \mathbb{R} \end{cases} .$$

THERMAL CONDUCTION PROBLEM

Weak form of the problem

Find u so that

$$\begin{cases} a\left(\upsilon,\varphi\right) = l\left(\varphi\right), & \forall \varphi \in H_0^1\left(\Omega\right),\ \upsilon \in H_0^1\left(\Omega\right) \\ \upsilon\left(\boldsymbol{x}\right) = 0, & \forall \boldsymbol{x} \in \Gamma_D \\ \gamma\left(\boldsymbol{x}\right) = g_N\left(\boldsymbol{x}\right), & \forall \boldsymbol{x} \in \Gamma_D,\ g_D : \Gamma_D \to \mathbb{R} \\ \kappa\left(\boldsymbol{x}\right) \dfrac{\partial\left(\upsilon + \gamma\right)}{\partial \boldsymbol{\nu}}\left(\boldsymbol{x}\right) = g_N\left(\boldsymbol{x}\right), & \forall \boldsymbol{x} \in \Gamma_N,\ g_N : \Gamma_N \to \mathbb{R} \\ u\left(\boldsymbol{x}\right) = \upsilon\left(\boldsymbol{x}\right) + \gamma\left(\boldsymbol{x}\right), & \forall \boldsymbol{x} \in \Omega \end{cases},$$

$$a\left(\upsilon,\varphi\right) = \iint_\Omega \kappa \nabla \upsilon \nabla \varphi d\boldsymbol{x},$$

$$l\left(\varphi\right) = \iint_\Omega \left(f\varphi - \kappa \nabla \gamma \nabla \varphi\right) d\boldsymbol{x} - \int_{\Gamma_N} \kappa g_N \varphi d\boldsymbol{\Gamma}.$$

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GALERKIN METHOD

- Approximate the space: $V = H_0^1(\Omega) \rightsquigarrow V_{h,p}$.
- We rewrite the equation $a(\upsilon, v) = l(v)\,,\ \forall v \in V$ with

  $a(\upsilon_{h,p}, v) = l(v)\,,\ \forall v \in V_{h,p} = \text{span}\left\{v_n, n = 0, \ldots, N_{h,p} - 1\right\} \subset V$

  where $\upsilon(x) = \sum_{i=0}^{N_{h,p}-1} \bar{\upsilon}_i v_i$.
- In this space any function can be written as the linear combination of the basis functions as

$$a\left(\sum_{i=0}^{N_{h,p}-1} \bar{\upsilon}_i v_i, v_j\right) = \sum_{i=0}^{N_{h,p}-1} a(v_i, v_j)\,\bar{\upsilon}_i = l(v_j)\,, j = 0, \ldots, N_{h,p}-1.$$

- The result is a linear system of equations which can be written in the matrix form $\boldsymbol{S}_{h,p}\bar{\boldsymbol{\Upsilon}}_{h,p} = \boldsymbol{F}_{h,p}$.

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ISOPARAMETRIC CONCEPT

Both Isogeometric Analysis and the Finite Element Method are based on the

Isoparametric concept

The same basis functions used to define the geometry are used to define the approximated solution.

FINITE ELEMENT METHOD (1)

Step 1: *Approximation* of the domain with a *piecewise-polynomial* boundary.

Step 2: The approximated domain is subdivided into N_{el} elements forming the *mesh*.

Step 3: As a consequence of the approximation, the domains of the functions defining the boundary conditions need to be modified.

Step 4: The space V of the solutions has to be redefined with a new space $V_{h,p}$ where the basis functions are associated with the nodes of the elements and are piecewise-polynomials of degree p.

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FINITE ELEMENT METHOD (2)

Step 5: The weak formulation can be rewritten using the approximations presented so far:

$$\sum_{i=0}^{N_{h,p}-1} \bar{v}_i \iint_{\Omega_h} \kappa \nabla v_i \nabla v_j d\mathbf{x} =$$

$$\iint_{\Omega_h} \left(f v_j - \kappa \nabla \gamma_{h,p} \cdot \nabla v_j \right) d\mathbf{x} + \int_{\Gamma_{N,h}} \kappa g_{N,h} v_j d\mathbf{S},$$

$j = 0, \ldots, N_{h,p} - 1.$

ISOPARAMETRIC ELEMENTS

Definition

Let $\boldsymbol{x}_{K_m} : \tilde{K} \to K_m$ be of the form

$$\boldsymbol{x}_{K_m}(\boldsymbol{\xi}) = \sum_{i=0}^{N_{h,p}-1} v_{\tilde{K}}^{i} \boldsymbol{x}_i^{(m)}$$

where $\boldsymbol{x}_i$ is the i^{th} node of the element $\mathcal{K}_m$. If the element interpolation function can be written as

$$\upsilon_{h,p}(\boldsymbol{\xi}) = \sum_{i=0}^{N_{h,p}-1} v_{\tilde{K}}^{i} \bar{\upsilon}_i$$

the element is said to be isoparametric.

RESULTING MODEL (1)

The *left-hand side* of the equation of the linear system can be written as:

$$\sum_{i=0}^{N_{h,p}-1} \bar{v}_i \sum_{m=1}^{M_{h,p}} \iint_{K_m} \kappa \nabla v_j \nabla v_i d\mathbf{x} =$$

$$= \sum_{j=0}^{N_{h,p}-1} \bar{v}_j \sum_{m=1}^{M_{h,p}} \sum_{n=1}^{2} \iint_{\tilde{K}} J_{K_m} \tilde{\kappa}^{(m)} \cdot \left(\sum_{r=1}^{2} \frac{\partial \tilde{v}_j^{(m)}}{\partial \xi_r} \cdot \frac{\partial \xi_r^{(m)}}{\partial x_n} \right) \cdot \ldots$$

$$\ldots \cdot \left(\sum_{s=1}^{2} \frac{\partial \tilde{v}_i^{(m)}}{\partial \xi_s} \cdot \frac{\partial \xi_s^{(m)}}{\partial x_n} \right) d\boldsymbol{\xi}.$$

where:

$$\left\{ \frac{\partial \xi_r^{(m)}}{\partial x_n} \right\}_{r,n=1}^{2} = \left(\frac{D\mathbf{x}_{K_m}}{D\boldsymbol{\xi}} \right)^{-1}.$$

RESULTING MODEL (2)

The *right-hand side* turns out to be:

$$\sum_{m=1}^{M_{h,p}} \iint_{K_m} \left(f v_j - \kappa \nabla \gamma_{h,p} \nabla v_j \right) d\mathbf{x} + \sum_{m=1}^{M_{h,p}} \int_{\Gamma_{N,h} \cap \bar{K}_m} \kappa g_{N,h} v_j d\mathbf{\Gamma} =$$

$$\sum_{m=1}^{M_{h,p}} \iint_{\tilde{K}} J_{K_m} \left(\tilde{f}^{(m)} \tilde{v}_j^{(m)} - \tilde{\kappa}^{(m)} \nabla \tilde{\gamma}_{h,p}^{(m)} \nabla \tilde{v}_j^{(m)} \right) d\boldsymbol{\xi} +$$

$$+ \sum_{m=1}^{M_{h,p}} \int_{\bar{\tilde{K}} \cap \tilde{\Gamma}_{N,h}} \left(\left(a_1 g_{N,h} v_j \right) \circ \mathbf{y}_{\bar{K}_m \cap \Gamma_{D,h}} \right) (t)\, dt.$$

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B-SPLINES basis functions (1)

Definition

A B-spline basis functions is defined as:

$$N_i^0(\xi) = \begin{cases} 1, & \xi_i \le \xi < \xi_{i+1} \\ 0, & \text{otherwise} \end{cases} \quad ,$$

$$N_i^p(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} \cdot N_i^{p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} \cdot N_{i+1}^{p-1}(\xi),$$

where $\Xi = [\xi_0, \ldots, \xi_m]$ is the knot vector.

B-SPLINES basis functions (2)

Definition

A knot vector of the form

$$\Xi = \left[\underbrace{a, \dots, a}_{p+1}, \xi_{p+1}, \dots, \xi_{m-p-1}, \underbrace{b, \dots, b}_{p+1} \right]$$

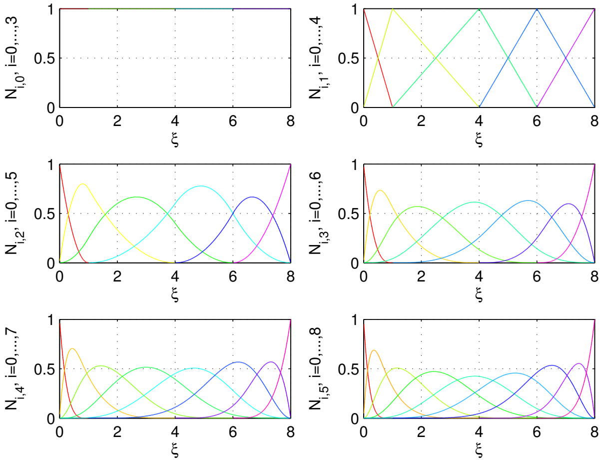
where p is the degree and m is the number of elements of the knot vector, is said to be *non-periodic* or *clamped* or *open*.

Derivatives of B-spline basis functions:

$$N_i^{p,(k)}(\xi) = p \left(\frac{N_i^{p-1,(k-1)}(\xi)}{\xi_{i+p} - \xi_i} - \frac{N_{i+1}^{p-1,(k-1)}(\xi)}{\xi_{i+p+1} - \xi_{i+1}} \right).$$

B-SPLINES basis functions (3)

Example of basis functions of degrees $p = 0, 1, 2, 3, 4, 5$



B-SPLINES

Definition

B-spline *curve*:

$$\boldsymbol{C}(\xi) = \sum_{i=0}^{n} N_i^p(\xi)\,\boldsymbol{P}_i, \;\; a \leq \xi \leq b.$$

Definition

B-spline *surface*:

$$\boldsymbol{S}(\xi,\eta) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_i^p(\xi)\,N_j^q(\eta)\,\boldsymbol{P}_{i,j}, \;\; \boldsymbol{a} \leq \boldsymbol{\xi} \leq \boldsymbol{b}.$$

NURBS basis functions (1)

Definition

A NURBS basis function is defined as:

$$R_i^p(\xi) = \frac{N_i^p(\xi)\, w_i}{\sum_{\hat{i}=0}^{n} N_{\hat{i}}^p(\xi)\, w_{\hat{i}}}, \quad \boldsymbol{a} \leq \boldsymbol{\xi} \leq \boldsymbol{b},$$

$$R_{i,j}^{p,q}(\xi,\eta) = \frac{N_i^p(\xi)\, N_j^q(\eta)\, w_{i,j}}{\sum_{\hat{i}=0}^{n}\sum_{\hat{j}=0}^{m} N_{\hat{i}}^p(\xi)\, N_{\hat{j}}^q(\eta)\, w_{\hat{i},\hat{j}}}, \quad \boldsymbol{a} \leq \boldsymbol{\xi} \leq \boldsymbol{b}.$$

NURBS basis functions (2)

Derivatives of NURBS basis functions:

$$\frac{\partial R_i^p(\xi)}{\partial \xi} = \frac{w_i \dfrac{\partial N_i^p(\xi)}{\partial \xi} \cdot W(\xi) - N_i^p(\xi)\, w_i \cdot \displaystyle\sum_{\hat{i}=0}^{n} \dfrac{\partial N_{\hat{i}}^p(\xi)}{\partial \xi} w_{\hat{i}}}{(W(\xi))^2},$$

$$W(\xi) = \sum_{\hat{i}=0}^{n} N_{\hat{i}}^p(\xi)\, w_{\hat{i}}.$$

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NURBS *surface*:

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NURBS'S (2)

Definition

NURBS *solid*:

$$\boldsymbol{S}(\xi,\eta,\zeta) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{l} R_{i,j,k}^{p,q,l}(\xi,\eta,\zeta)\,\boldsymbol{P}_{i,j,k},\ \ \boldsymbol{a} \le \boldsymbol{\xi} \le \boldsymbol{b}.$$

T-SPLINES

- *Generalization* of NURBS's;
- *local refinement*;
- *backwards-compatible* with NURBS;

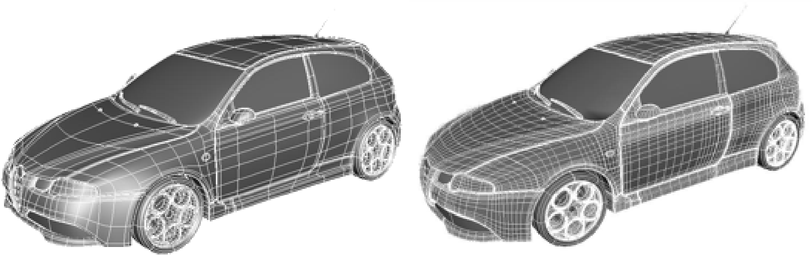

Figure: On the left car model designed using T-splines, on the right the same exact model is converted to a NURBS-based model.

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OVERVIEW OF ISOGEOMETRIC ANALYSIS

- The mesh is defined by the *product of knot vectors*.
  - Knot spans subdivide the *patch* into elements.
- The *support* of the basis functions is small (just like basis functions used in FEM). This leads to *sparse matrices*.
- The *isoparametric concept* is invoked: the solution field is defined by using the same basis functions used to describe the geometry.
  - The *DOFs* are the control points.
- *Boundary conditions* are applied like in FEM.

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TRANSFORMATION TO THE PARAMETRIC SPACE

According to the isoparametric concept, the same basis functions used for the geometry (we used B-splines for instance) are used to build the *solution field*. Therefore, it can be expressed with

$$\upsilon_h(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} N_{i,j}^{p,q}(x,y)\,\bar{\upsilon}_{i,j},$$

and the *geometrical map* used to map points from the physical space to the parametric space is:

$$\tilde{\boldsymbol{x}}(\xi,\eta) = \sum_{i=0}^{n}\sum_{j=0}^{m} \tilde{N}_{i,j}^{p,q}(\xi,\eta)\,\boldsymbol{P}_{i,j} \in \mathcal{S}_{\Xi,H}^{p,q},$$

where $\boldsymbol{\xi} = [\xi,\eta]^T \in \tilde{\Omega} = [0,1]\times[0,1] \subset \mathbb{R}^2$.

$\{h, p, hp, ph\}$-REFINEMENTS

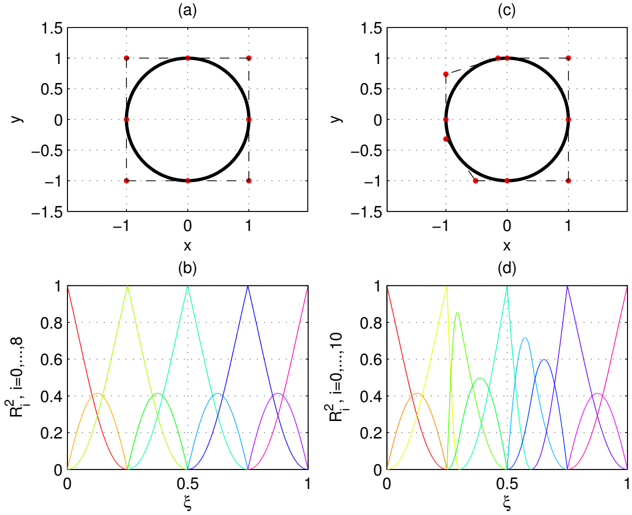
h-refinement: this kind of refinement is performed by the knot insertion procedure;

p-refinement: this kind of refinement is performed in IGA by the degree elevation procedure;

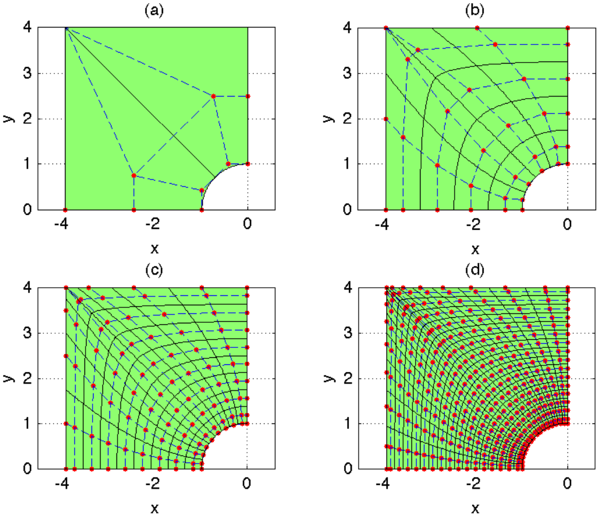
hp-refinement: this kind of refinement is performed by by knot insertion followed by degree elevation;

ph-refinement: this kind of refinement is performed by degree elevation followed by knot insertion.

EXAMPLE OF *h*-REFINEMENT ON CURVES



EXAMPLE OF *h*-REFINEMENT ON SURFACES



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NUMERICAL EXAMPLE (1)

Example

Example of solutions using FEM and IGA of the problem

$$\begin{cases} \nabla\left(\nabla u\left(\boldsymbol{x}\right)\right)=1, & \forall \boldsymbol{x}\in\Omega,\ \text{given } u:\bar{\Omega}\to\mathbb{R} \\ u\left(\boldsymbol{x}\right)=0, & \forall \boldsymbol{x}\in\partial\Omega \end{cases},$$

where the domain is first a square plate, and then the same plate with a hole in a corner.

SOLUTION ON SQUARE PLATE

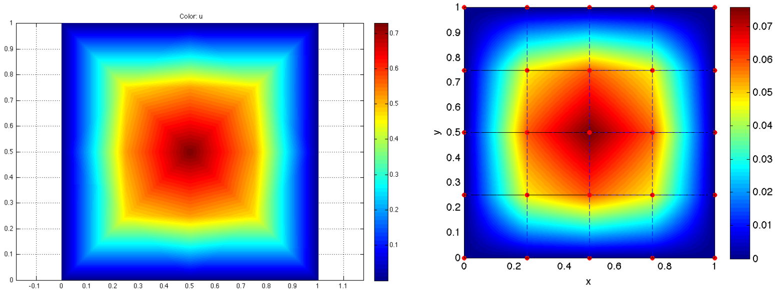
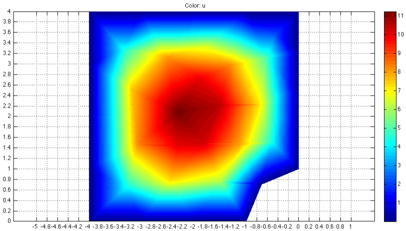
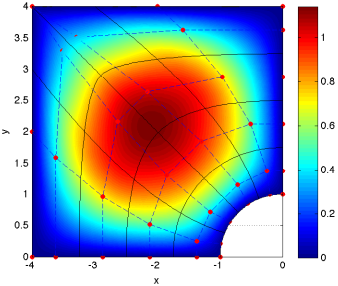
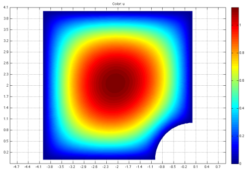


Figure: Solution using FEM with 25 nodes on the left, IGA with 25 nodes on the right.

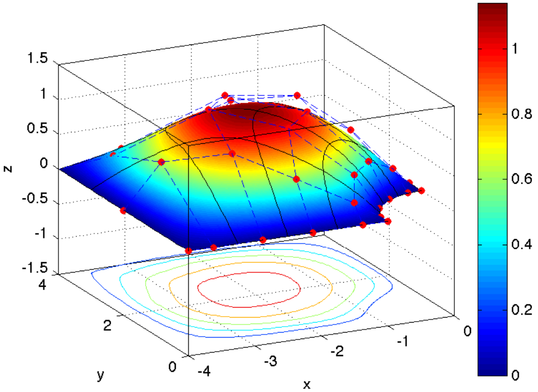
SOLUTION ON SQUARE PLATE WITH HOLE







SOLUTION ON SQUARE PLATE WITH HOLE WITH IGA



Outline

1. Introduction
   - Automated design process
   - Partial Differential Equations
   - Galerkin method
   - Isoparametric concept
2. Finite Element Method
3. Isogeometric Analysis
   - CAD basis functions
   - Analysis
4. Numerical examples
   - Numerical example (1)
   - Numerical example (2)
   - Numerical example (3)

DEFINITION OF THE PROBLEM

Example

Example of solutions using FEM and IGA of the problem

$$\begin{cases} \nabla\left(\nabla u\left(\boldsymbol{x}\right)\right) = 1, & \forall \boldsymbol{x} \in \Omega, \text{ given } u : \bar{\Omega} \to \mathbb{R} \\ u\left(\boldsymbol{x}\right) = 0, & \forall \boldsymbol{x} \in \Gamma_{D,0} = \left\{\boldsymbol{x} | \tilde{\boldsymbol{x}}\left(\xi,\eta\right) = \boldsymbol{x}, \xi = 0, \eta \in [0,1]\right\} \\ u\left(\boldsymbol{x}\right) = 1, & \forall \boldsymbol{x} \in \Gamma_{D,1} = \left\{\boldsymbol{x} | \tilde{\boldsymbol{x}}\left(\xi,\eta\right) = \boldsymbol{x}, \xi = 1, \eta \in [0,1]\right\} \\ u\left(\boldsymbol{x}\right) = 2, & \forall \boldsymbol{x} \in \Gamma_{D,2} = \left\{\boldsymbol{x} | \tilde{\boldsymbol{x}}\left(\xi,\eta\right) = \boldsymbol{x}, \xi \in [0,1], \eta = 1\right\} \\ \dfrac{\partial u}{\partial \boldsymbol{\nu}} = 9x, & \forall \boldsymbol{x} \in \Gamma_N = \left\{\boldsymbol{x} | \tilde{\boldsymbol{x}}\left(\xi,\eta\right) = \boldsymbol{x}, \xi \in [0,1], \eta = 0\right\} \end{cases} \tag{1}$$

where the domain is first a square plate, and then the same plate with a hole in a corner.

SOLUTION ON SQUARE PLATE

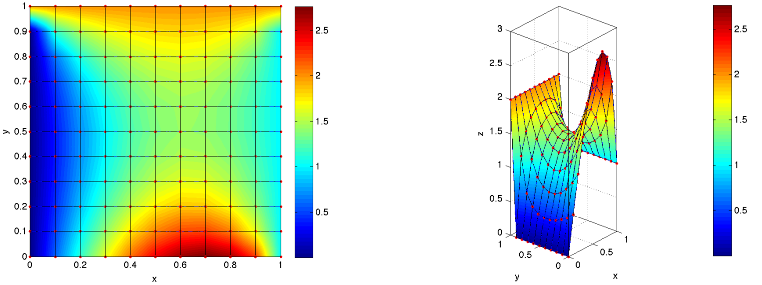


Figure: Solution using 121 nodes.

SOLUTION ON SQUARE PLATE WITH HOLE

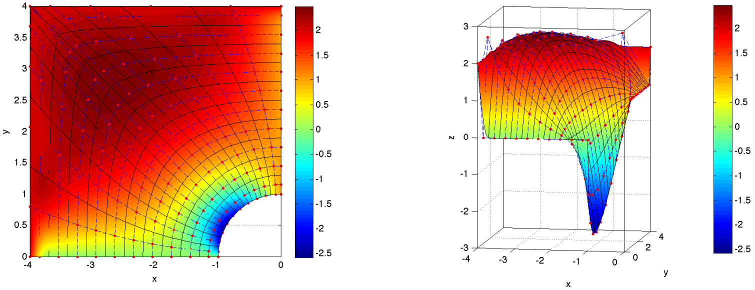


Figure: Solution using 144 nodes.

Outline

1. Introduction
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DEFINITION OF THE PROBLEM

Example

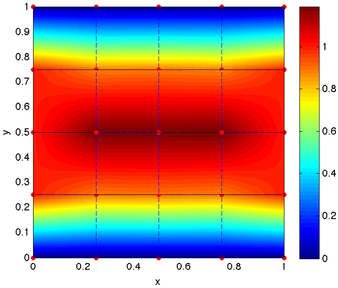
Example of solutions using FEM and IGA of the problem

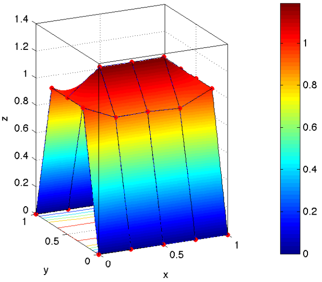
$$\begin{cases} \nabla\left(2x\mathbf{I}_2\nabla u\left(\boldsymbol{x}\right)\right) = 20x, & \forall \boldsymbol{x} \in \Omega \\ u\left(\boldsymbol{x}\right) = 1, & \forall \boldsymbol{x} \in \Gamma_D \end{cases},$$

where the domain is first a square plate, and then the same plate with a hole in a corner.

h-REFINEMENT BY KNOT INSERTION (9 NODES)

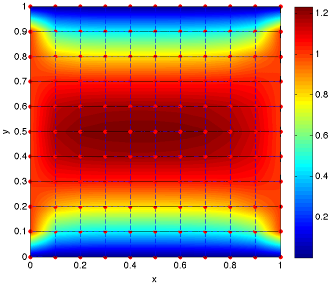
h-refinement on square plate

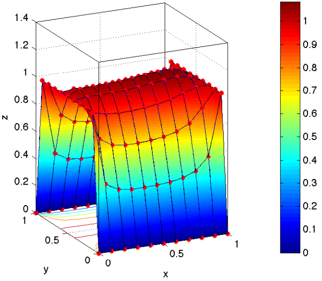




h-REFINEMENT BY KNOT INSERTION (81 NODES)

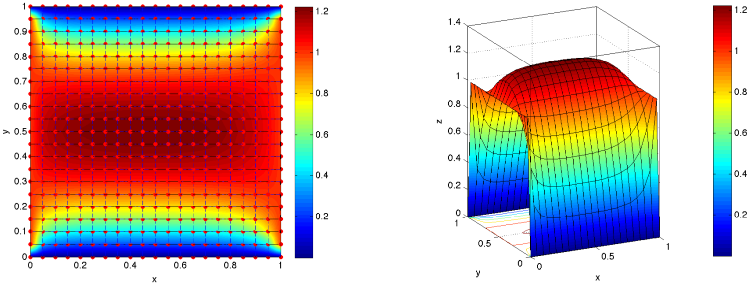
h-refinement on square plate





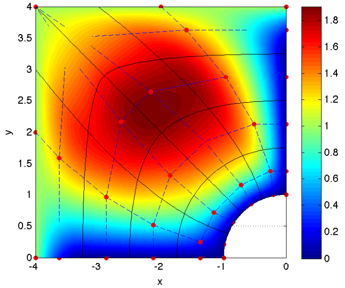
h-REFINEMENT BY KNOT INSERTION (361 NODES)

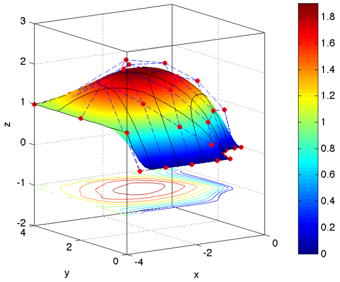
h-refinement on square plate



h-REFINEMENT BY KNOT INSERTION (16 NODES)

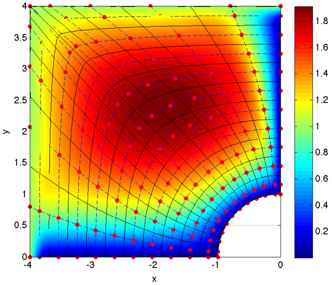
h-refinement on square plate with hole

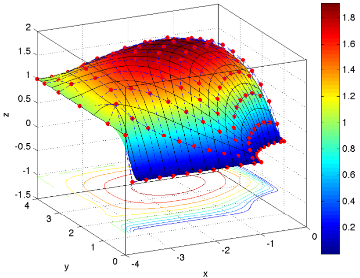




h-REFINEMENT BY KNOT INSERTION (100 NODES)

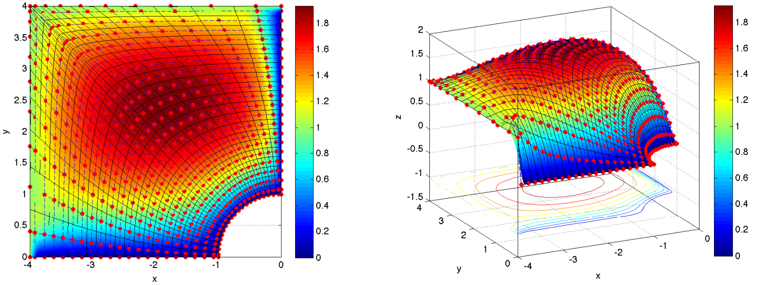
h-refinement on square plate with hole





h-REFINEMENT BY KNOT INSERTION (400 NODES)

h-refinement on square plate with hole



END

THANK YOU FOR YOUR ATTENTION!

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